2965. Proposed by C. N. MILLS, Tiffin, Ohio.

If a quadrilateral inscribed in a square has the diagonals a and b, and the area A, show that the area of the square is $\dfrac{a^2b^2 - 4A^2}{a^2 + b^2 - 4A}$.

SOLUTIONS

2824 [1920, 185]. Proposed by G. Y. SOSNOW, Newark, N. J.

If n_1, n_2, n_3, n_4 be the lengths of the four normals and t_1, t_2, t_3 the lengths of the three tangents drawn from any point to the semi-cubical parabola, $ay^2 = x^3$, then will $27n_1n_2n_3n_4 = at_1t_2t_3$ [From *Mathematical Tripos Examination*, Cambridge, England].

SOLUTION BY J. B. REYNOLDS, Lehigh University.

Let the parametric equations of the curve be

$$x = au^2 \qquad \text{and} \qquad y = au^3.$$

Then

$$\frac{dy}{dx} = \frac{3u}{2},$$

the equations of the normal and tangent will be

$$3au^4 + 2au^2 - 3yu - 2x = 0 \tag{1}$$

and

$$au^3 - 3xu + 2y = 0, \tag{2}$$

and the lengths of the normal and tangent from (x, y) to the curve will be

$$n = \left(\frac{x - au^2}{3u}\right)\sqrt{9u^2 + 4} \qquad \text{and} \qquad t = \left(\frac{x - au^2}{2}\right)\sqrt{9u^2 + 4}.$$

The polynomial whose roots are the squares of the roots of (1) is easily found[1] to be:

$$9a^2z^4 + 12a^2z^3 + 4a(a - 3x)z^2 - (8ax + 9y^2)z + 4x^2 = 9a^2\Pi(z - u_i^2).$$

Multiply the roots by a, according to the familiar rule, and then replace z by x. This gives

$$\Pi(x - au_i^2) = x(x^3 - ay^2).$$

Also multiplying the roots by 9 and replacing z by -4 we obtain

$$\Pi(4 + 9u_i^2) = \frac{4}{a^2}[729(x^2 + y^2) + 216ax + 16a^2].$$

Hence

$$n_1n_2n_3n_4 = -\frac{x^3 - ay^2}{27}[729(x^2 + y^2) + 216ax + 16a^2]^{1/2}.$$

In a similar manner from equation (2) we get,

$$\Pi(x - au_i^2) = 4(x^3 - ay^2),$$

$$\Pi(4 + 9u_i^2) = \frac{4}{a^2}[729(x^2 + y^2) + 216ax + 16a^2];$$

and hence,

$$t_1t_2t_3 = \frac{x^3 - ay^2}{a}[729(x^2 + y^2) + 216ax + 16a^2]^{1/2}.$$

Therefore

$$at_1t_2t_3 = 27n_1n_2n_3n_4$$

neglecting the sign.

2834 [1920, 273]. Proposed by OTTO DUNKEL, Washington University.

In any triangle ABC let M and N be, respectively, the points in which the median and the bisector of the angle at A meet the side BC, Q and P the points in which the perpendicular at N to NA meets MA and BA, respectively, and O the point in which the perpendicular at P to

[1] See, for example, Todhunter, *An Elementary Treatise on the Theory of Equations*, London, 1880, p. 36; or Salmon, *Lessons Introductory to the Modern Higher Algebra*, Dublin, 1885, p. 350.—EDITORS.